The Micromechanics of Colloidal Dispersions

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Colloid Dispersions as ‘Soft Matter’

- Many complex fluids are composed of (or can be modeled as) suspensions of small particles dispersed in a viscous fluid where Brownian forces ($kT$) compete against interparticle ($V$) and hydrodynamic ($Re << 1$) forces to set structure and determine properties.

- What makes colloidal fluids interesting is that they are ‘soft’

$$\text{colloidal} \sim \frac{(a_x)^3}{a_p} ; \text{ time scales} \sim \frac{(a_p)^3}{a_x}$$

- And often far from equilibrium
Micromechanics ($Re << 1$)

Particle Motion: 

\[ m \frac{dU'}{dt} = F'^{\text{H}} + F'^{\text{B}} + F'^{\text{E}} \]

Hydrodynamic: 

\[ F'^{\text{H}} = -\mathbf{R}(x) \left( U \cdot U' \right) \]

Stokes drag

Brownian: 

\[ F'^{\text{B}} = 0 \quad F'^{\text{B}}(0) F'^{\text{B}}(t) = 2kT \mathbf{R}(x) \delta(t) \]

\[ \mathcal{O}(10^{-3} \text{ s}) \]

Interparicle/external: 

\[ F'^{\text{E}} = \Delta \rho V_{i} \mathbf{g}, \text{ electrostatic, etc.} \]

Fluid Motion: 

Stokes Equations

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \mathbf{u} = U + x \times \Omega \]

no slip at particle surfaces
Add Shearing Motion

Grand Resistance Matrix

\[ U^- - U^+ + \dot{\mathbf{1}} \cdot X, \quad \dot{\mathbf{1}} = \mathbf{\nabla} u - \Omega^2 + E^2 \]

\[
\begin{pmatrix}
F^{ii} \\
L^{ii} \\
S^{ii}
\end{pmatrix} - \begin{pmatrix}
R_{i1} & R_{i2} & R_{i3} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix} \cdot \begin{pmatrix}
\mathbf{U} \cdot \mathbf{U}^* \\
\Omega \cdot \Omega^* \\
\mathbf{E}^*
\end{pmatrix}
\]

Hydrodynamic Stresslet: \[ S^{ii} = \frac{1}{2} \int (\rho \mathbf{\sigma} \cdot \mathbf{n} + \sigma \cdot \mathbf{mp}) dS \]
Many particle hydrodynamics - Geometry Only!

\[
\left( \begin{array}{c}
F^{c'} \\
F^{d'} \\
F^{e'} \\
\vdots \\
\end{array} \right) - \left( \begin{array}{cccc}
R^{c'c} & R^{c'd} & R^{c'e} & \cdots \\
R^{d'c} & R^{d'd} & R^{d'e} & \cdots \\
R^{e'c} & R^{e'd} & R^{e'e} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array} \right) \left( \begin{array}{c}
U^{c} \\
U^{d} \\
U^{e} \\
\vdots
\end{array} \right)
\]

\[
R^{c'i} \left( r_{c'i} r_{d'j} r_{e'k} \cdots ; a_c, a_d, a_e, \cdots \right)
\]

In short hand: \[
F^{ij} = \cdot R^{ji} \cdot U^i
\]

Many-particle grand resistance tensor
Stokesian Dynamics: $F^H = - R(x) \cdot U$

Periodic Boundaries

Implement matched asymptotic expansions dynamically for thousands of particles in $O(N)$ operations for millions of time steps.

$F'' = \cdot R^\ast(x) \cdot U$

$\frac{dx}{dt} = U = (R^\ast(x))^{-1} \cdot F_{net}$
Macroscopic Properties

• How are they defined?

• How computed?

• Need both interactions (hydrodynamic and interparticle) \textit{AND} microstructure.
Macroscopic Properties

- **Self-Diffusion (Stokes-Einstein-Sutherland)**
  \[ D = kT \left( \frac{1}{R_{11}} \right) - kT \left( \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{R_{ii}} \right) \right) \]

  **Note:**
  \[ (R_{ii})_{mm} = (\langle R_{ii} \rangle)_{mm} \]

- **Sedimentation**
  \[ \langle U \rangle - \langle R_{12} \rangle \cdot F = \left( \frac{1}{N} \sum_{i=2}^{N} (R_{i1})_{mm} \right) \cdot F \]

- **Permeability (porous media)**
  \[ \langle F^{\infty} \rangle - \langle R_{11} \rangle \cdot U^{\infty} = \eta K - \left( \frac{1}{N} \sum_{i=1}^{N} R_{i1} \right) \]

\[ \left( \sum_{i=1}^{N} R_{i1} \right)^{1} \]
Macroscopic Properties: Rheology

Bulk Stress:

\[ \langle \Sigma \rangle = \langle \rho \rangle J + 2\eta \langle \epsilon \rangle \cdot \eta (\Sigma) \]

\[ \langle \Sigma_{\nu} \rangle = \langle \nu_{\alpha} \cdot \nu_{\beta} \cdot \nu_{\gamma} \cdot \nu_{\delta} \rangle \cdot \langle \epsilon \rangle \]

\[ \cdot \left( \langle \nu_{\alpha} \cdot \nu_{\beta} + X' J \rangle \cdot F' \rangle \cdot \epsilon \cdot \langle L' \rangle \right) \]

\[ \cdot kT J \cdot kT (\nu \cdot \nu_{\alpha} \cdot \nu_{\beta}) \]
Interparticle forces

Steric Stabilization

Electrostatic Stabilization

DLVO Theory
Macroscopic Properties: Rheology

Bulk Stress:

\[
\langle \Sigma \rangle = \langle \rho \rangle I + 2\eta \langle \epsilon \rangle \cdot \sigma \langle \Sigma \rangle
\]

\[
\langle \Sigma_\nu \rangle = \langle R_{\nu \mu} \cdot R_{\nu \mu} \cdot R_{\nu \mu} \cdot R_{\nu \mu} \rangle \cdot \epsilon \cdot \langle L \rangle
\]

\[
= kT J \cdot kT \langle \nabla \cdot R_{\nu \mu} \cdot R_{\nu \mu} \rangle
\]

number density \( \rho \)

Shear

Interparticle

Brownian
Equilibrium Osmotic Pressure

\[ \Pi^{osm} = n k T + k T \langle \nabla \cdot R_{SU} \cdot R_{FU}^{-1} \rangle : I / 3 \]
Brownian Motion: Diffusion

Stokes-Einstein-Sutherland Relation

\[ D = kT M \left( = \frac{kT}{6\pi \eta a} \right) \]

\[ F'' = 6\pi \eta a U \]

\[ Re = \frac{\rho U a}{\eta} \ll 1 \]
Short-time self-diffusivity

\[ D_s(\phi) = kT \langle M \rangle \]

Dilute limit: \( \phi \to 0 \)

\[ D_s(\phi) \sim D_0 \left( 1 - 1.83\phi \right) \]

Close packing:

\[ c = \frac{1}{2} - \phi, \phi_{\text{c}} \to 0 \]

\[ D_s(\phi) \sim D_0 / \ln(1/c) \]

Sierou & Brady *JFM* (2001)
Near Equilibrium Behavior:

\[ \eta' \sim 1 + \frac{5}{3} \phi + 5 \phi^3 \quad \text{as} \quad \phi \to 0 \]

\[ \eta' \sim \ln(1 - \phi/\phi_{rcp}) \quad \text{as} \quad \phi \to \phi_{rcp} \]
High-frequency dynamic viscosity & short-time self-diffusivity
High-Frequency Elastic Modulus

Stokesian Dynamics

Shikata & Pearson (1994)

\( G'_{zd} / kT \)

\( \phi = \phi_m \)
Brownian Self-Diffusivity (long-time)

The self-diffusivity decreases with increasing concentration as the diffusing particle must push past its neighbors to move.

Brady *JFM* (1994)

Experimental Results
- Van Veluwen & Lekkerker (1988)
- Ottewill & Williams (1987)
- Van Megen, Underwood & Snook (1986)
- Kops-Werkhoven & Fijnaut (1982)

Simulation Results
- Phung (1993); N=27; Pe=0.01
- Cichocki & Hinsen (1990) x $D_0(\phi)/D_0$

Theoretical Prediction

Mode Coupling: $D'_i(\phi) \sim D'_0(\phi) \cdot \phi^{\nu}$

Fuchs *et al* (1992)

Uniform silica microspheres (1 μm diam., ×10,000).
Zero-shear Brownian viscosity ($Pe = 0$)

\[ \Delta \eta'' = \eta(Pe \to 0) \cdot \eta' \]

\[ \eta = \eta' \phi + \int_0^\phi \langle \nabla \cdot \mathbf{R}_{ij} \mathbf{R}_{ij}(0) \mathbf{V} \cdot \mathbf{R}_{ij} \mathbf{R}_{ij} \rangle \, dt \]

(Banchio & Brady 2003)

Pusey & van Megen (1986)

\[ 0.48 \quad 0.50 \quad 0.55 \quad \cdots \quad 0.58 \quad \cdots \quad 0.61 \]
Steady shear viscosity & long-time self-diffusivity
Gradient or collective or mutual-diffusivity

Batchelor *JFM* (1976)

\[
D = \frac{K(\phi)}{6\pi \eta a} \left[ \phi \left( \frac{\partial \mu}{\partial \phi} \right)_f \right] 
\]

\[K(\phi) = \text{hindered settling function}\]

\[\mu = \text{chemical potential}\]

Dilute limit, \( \phi \to 0 \)

\[D \sim D_i (1 - 1.45\phi)\]
Particle Diffusion: \[ D \sim (v')^2 \times \tau \]

- Brownian motion

\[ (v')^2 \sim \frac{3kT}{m}, \quad \tau \sim \frac{m}{6\pi \eta a}, \quad D \sim \frac{kT}{2\pi \eta a} \]

- Time to diffuse the order of the particle size

\[ t_\alpha \sim a' \quad D \sim \frac{2\pi \eta a'}{kT} \quad = \ 1s \ \text{for} \ a = 0.5\mu m \]
The self-diffusivity decreases with increasing concentration as the diffusing particle must push past its neighbors to move.

Brady (1994)
Particle Diffusion: \[ D \sim (v')^2 \times \tau \]

- **Brownian motion**

  \[ (v')^2 \sim \frac{3kT}{m}, \quad \tau \sim \frac{m}{6\pi\eta a}, \quad D \sim \frac{kT}{2\pi\eta a} \]

  \[ t_e \sim \frac{a}{D} \sim \frac{2\pi\eta a}{kT} \]

  - 1s for \( a = 0.5\mu m \)
  - 1000s for \( a = 5\mu m \)
  - \( 10^9 s \sim 30\text{ yrs} \) for \( a = 500\mu m \)

- **Shear-induced:** \( Pe = \frac{\dot{\gamma}a}{D_p} \gg 1 \)

  \[ (v')^2 \sim (\dot{\gamma}a)^2, \quad \tau \sim \dot{\gamma}^{-1}, \quad D \sim \dot{\gamma}a^2 \]
GI Taylor film: “Low Reynolds Number Flows”
Does shear-induced diffusion exist? You Judge!

Run A

View is in the velocity-gradient -- vorticity plane

Run B

Simple Shear Flow
\( \nabla \cdot v = 0.35, \ Re \ll 1 \)

\( Pe = 0 \)
Which one is which?

Run C

Same as previous runs, but at a finer time scale

Run D
Oscillatory Shear: Chaos & Reversibility


Small Strain Amplitude

Large Strain Amplitude
Oscillatory Shear: Lyapunov Exponent

\[ \lambda = \left. \frac{d}{dt} \ln \left[ \frac{d(t)}{d(0)} \right] \right|_{t=0} \]

Relaxation to steady state  Separation in phase space
Oscillatory Shear: Diffusivities
Shear-Induced Diffusivity

Stokesian Dynamics
\( \phi = 0.45 \)

\( D_{xx}, D_{yy}, D_{zz} \)

\[ D(Pe) = \frac{D}{D_0} \]

\[ Pe = \frac{\gamma a^2}{D_0} = \frac{6\pi \eta a^2 \gamma}{kT} \]
Shear-Induced Self-Diffusivity

\[ \frac{D_{yy}}{\gamma^2 \Omega} \]

\( \phi \)

Leighton & Acrivos (1987)
Eckstein et al. (1977)
Phan & Leighton (1999)
Breedveld et al. (1998)
Breedveld et al. (2000)
Foss & Brady (1999)
Sierou & Brady (2001)
Long-time Asymptote?

\[ \gamma = 0.2 \]

**Short times**

**Long times**
Simulation vs. Experiment (small strains)
Shear-Induced Self-Diffusivity

The shear-induced self-diffusivity increases with increasing concentration as particle collisions are responsible for the diffusive motion.
Diffusion Across Scales

Method
- Granular Dynamics
- Stokesian Dynamics
- Brownian Dynamics

Time Scale
- 1 hr
- 1 s
- 1 ns

Size Scale
- 1 nm
- 1 μm
- 1 mm

Method
- Stokesian Dynamics
- Brownian Dynamics
- Granular Dynamics

Time Scale
- Granular Media

Proteins

Colloids

Suspensions

Stokesian Dynamics

EXPERIMENTAL RESULTS
- Van Velzen & Lekkerkerker (1987)
- Oettel & Williams (1967)
- Van Meegen, Underwood, & Smook (1986)
- Kuyper-Weltboer & Fijten (1992)

THEORETICAL PREDICTION

$D \sim a^2$

$Pe \gg 1$, $St = 0$

$Pe \leq O(1)$
Particle Diffusion: \[ D \sim (v')^2 \times \tau \]

- **Brownian motion (Stokes-Einstein-Sutherland)**
  \[ (v')^2 \sim \frac{3kT}{m}, \quad \tau \sim \frac{m}{6\pi\eta a}, \quad D \sim \frac{kT}{2\pi\eta a} \]

- **Shear-induced**
  \[ (v')^2 \sim \dot{\gamma}a^2, \quad \tau \sim \dot{\gamma}^2, \quad D \sim \dot{\gamma}a^2 \]

- **Granular Gas (Kinetic Theory)**
  \[ (v')^2 \sim \tau_s, \quad \tau \sim \frac{\lambda}{v'(v')^2}, \quad D \sim \lambda \sqrt{\tau_s} \sim \gamma a \lambda \]
  \[ \tau_s \sim (\dot{\gamma}a)^2 \]
Micromechanics: Particle Inertia ($St \neq 0$)

Particle Motion:

$$m \cdot \frac{dU}{dt} = F^{\text{int}} + F^{\text{vis}} + F^{\text{ext}}$$

Hydrodynamic:

$$F^{\text{int}} = -\mathcal{R}(x) \cdot (U \cdot U')$$

Stokes drag

Interparicle/external:

$$F^{\text{vis}} = \text{Hard-sphere or inelastic collisions}$$

Dissipation:

$$e$$

Stokes number:

$$St = \gamma \frac{\dot{\gamma}}{6 \pi \eta a} \sim \rho \frac{\dot{\gamma} a^2}{\eta}$$

Reynolds number:

$$Re = \frac{\rho \dot{\gamma} a^2}{\rho \gamma}$$

Uniform silica microspheres (1 μm diam, × 10,000).
Self Diffusion vs. St

Savage & Dai (1993):

\[ D = \frac{a v \pi}{4(1 + e) \phi \xi(\phi)} \]

Energy balance:

(Sangani et al 1996)

\[ \eta \gamma^2 - \dot{\Phi}_{\text{viscous}} + \dot{\Phi}_{\text{inertial}} \]

\[ \dot{\Phi}_{\text{inertial}} \sim \eta a^2 \rho T^2 \]

\[ \dot{\Phi}_{\text{viscous}} \sim \rho a^2 \sqrt{(1 - \epsilon^2)} \phi T^2 \]

\[ \eta \sim \rho a^2 \sqrt{T_c} \]
Self Diffusion at finite $St$
Diffusion Across Scales

**Particle Diffusion:**

- **Brownian:** \( D \sim kT/\eta a \) \( \downarrow \) as \( \phi \) \( \uparrow \)
- **Viscous:** \( D \sim \eta a \) \( \uparrow \) as \( \phi \) \( \uparrow \)
- **Inertial:** \( D \sim \lambda \sqrt{I_s} \) \( \downarrow \) as \( \phi \) \( \uparrow \) for \( S_t \geq 10 \)
Shear-Induced Particle Diffusion

- We see that particles undergo a random walk and diffuse, even though there are no thermal effects.
- But what does this have to do with the phase behavior in two-phase flow?
- Diffusion usually smoothes out concentration variations and makes the system more homogeneous.
- But ...

\[ D \sim \dot{\gamma} a^3 d(\phi) \]

- If the shear rate varies in a flow, then particles will migrate to regions of low shear rate (Leighton & Acrivos 1987).
Particle migration in pressure-driven flow

C. Gao & J.G. Gilchrist, Lehigh

\[ D \sim \gamma a^2 d(\phi) \]

Lyon & Leal (1998)
Stress-induced diffusion

Steady 1D flow:

\[ \frac{\partial \sigma_{yy}}{\partial y} = 0 \Rightarrow \sigma_{yy} = \text{const.} \]

Dimensional analysis:

\[ a, \ H, \ \phi, \ \eta, \ \dot{\gamma} \]

\[ \therefore \sigma_{yy} \sim \eta \gamma f(\phi, a/H) \]

The particles go to the center!
Suspension Balance Model

Suspension: \[ \frac{\rho D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{\sigma} , \quad \nabla \cdot \mathbf{u} = 0 \]

Particles: \[ \rho_p \phi \frac{D\mathbf{u}_p}{Dt} = \nabla \cdot \mathbf{\sigma}_p - \mathbf{R}(\phi) \phi (\mathbf{u}_p - \mathbf{u}) , \quad \frac{\partial \phi}{\partial t} + \nabla \cdot \phi \mathbf{u}_p = 0 \]

\[ \frac{\partial \phi}{\partial t} + \nabla \cdot \phi \mathbf{u} = -\nabla \cdot \frac{1}{\mathbf{R}(\phi)} \nabla \cdot \mathbf{\sigma}_p \]

Stress-induced diffusion

Nott & Brady, JFM 1994
Suspension Balance Model

\[ \rho_p \phi \frac{D u_p}{D t} = \nabla \cdot \sigma_p - R(\phi) \phi (u_p - u), \quad \frac{\partial \phi}{\partial t} + \nabla \cdot \phi u_p = 0 \]

\[ \frac{\partial \phi}{\partial t} + \nabla \cdot \phi u = -\nabla \cdot \frac{1}{R(\phi)} \nabla \cdot \sigma_p \]

Equilibrium osmotic pressure

\[ \sigma_p^{eq} = -\Pi^{osm}(\phi) I \]

\[ \therefore \quad \frac{\partial \phi}{\partial t} = \nabla \cdot D^c \nabla \phi \]

\[ D^c = K(\phi) \frac{\partial \Pi^{osm}}{\partial \phi} = K(\phi) \frac{\phi}{1 - \phi} \frac{\partial \mu}{\partial \phi} \]
Suspension Balance Model

Suspension:
\[
\rho \frac{Du}{Dt} = \nabla \cdot \sigma , \quad \nabla \cdot u = 0 \quad \sigma = -p_f I + 2\eta e + \sigma_p
\]

Particles:
\[
\rho_p \phi \frac{Du_p}{Dt} = \nabla \cdot \sigma_p - R(\phi) \phi (u_p - u) , \quad \frac{\partial \phi}{\partial t} + \nabla \cdot \phi u_p = 0
\]
\[
\frac{\partial \phi}{\partial t} + \nabla \cdot \phi u = -\nabla \cdot \frac{1}{R(\phi)} \nabla \cdot \sigma_p
\]
\[
\sigma_p = -\Pi(\phi, Pe) I + 2\eta_p(\phi, Pe) e + N_p(\phi, Pe)
\]

pressure \quad shear viscosity \quad normal stress differences
Normal Stress Effects

Rod climbing ($N_1 > 0$)

(From DPL1, Bird, Armstrong & Hassager)

Rod falling ($N_2 + N_1/2 < 0$)


Another Issue: Flows with curved streamlines

Parallel Plate: \( \dot{\gamma} = \Omega \frac{r}{H} \)

Cone & Plate: \( \dot{\gamma} \) is constant


\[ a = 50 \text{ m} \]
\[ \gamma = 0.5 \]

before  

after  

\[ N_1 = \cdots \frac{\varepsilon}{a} N_2 \cdots \frac{\varepsilon}{a} I \]

\( \angle \) stress balance: no migration

\[ N_1 = \cdot 2N_2 \]
Suspension Balance Model

**Suspension:**

\[ \rho \frac{Du}{Dt} = \nabla \cdot \sigma , \nabla \cdot u = 0 \]

\[ \sigma = -p_f I + 2\eta_e + \sigma_p \]

**Particles:**

\[ \rho_p \phi \frac{Du_p}{Dt} = \nabla \cdot \sigma_p - R(\phi)\phi(u_p - u) , \frac{\partial \phi}{\partial t} + \nabla \cdot \phi u_p = 0 \]

\[ \frac{\partial \phi}{\partial t} + \nabla \cdot \phi u = -\nabla \cdot \frac{1}{R(\phi)} \nabla \cdot \sigma_p \]

\[ \sigma_p = -\Pi(\phi, Pe) I + 2\eta_p(\phi, Pe)e + N_p(\phi, Pe) \]

pressure \quad shear \text{ viscosity} \quad normal \text{ stress differences}
Modeling Suspension Flows (Fang et al 2002)

Eccentric journal bearing, \( \frac{e}{c} = 1/2, \quad \gamma = 0.5, \quad \frac{H}{a} = 35.3 \)

<table>
<thead>
<tr>
<th># turns</th>
<th>40</th>
<th>1000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Diffusive Flux Model</strong></td>
<td><img src="image1" alt="Diffusive Flux Model" /></td>
<td><img src="image2" alt="Diffusive Flux Model" /></td>
<td><img src="image3" alt="Diffusive Flux Model" /></td>
<td><img src="image4" alt="Diffusive Flux Model" /></td>
</tr>
<tr>
<td><strong>Suspension Balance Model</strong></td>
<td><img src="image5" alt="Suspension Balance Model" /></td>
<td><img src="image6" alt="Suspension Balance Model" /></td>
<td><img src="image7" alt="Suspension Balance Model" /></td>
<td><img src="image8" alt="Suspension Balance Model" /></td>
</tr>
<tr>
<td><strong>Experiment</strong></td>
<td><img src="image9" alt="Experiment" /></td>
<td><img src="image10" alt="Experiment" /></td>
<td><img src="image11" alt="Experiment" /></td>
<td><img src="image12" alt="Experiment" /></td>
</tr>
</tbody>
</table>
Particle Migration

- Because the diffusivity is proportional to the shear rate, when the shear rate varies as in pressure-driven flow, the particles migrate from regions of high shear rate to low, much as molecules migrate from high temperature to low -- the Soret effect.

- This behavior can be modeled by writing mass and momentum balances for the particles and fluid -- two-phase flow equations.
### Particle Size Scale

<table>
<thead>
<tr>
<th>Size</th>
<th>Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 m</td>
<td>$10^0$</td>
</tr>
<tr>
<td>1 mm</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>1 nm</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>1 Å</td>
<td>$10^{-10}$</td>
</tr>
</tbody>
</table>

### Simulation Method

- **Granular Dynamics** ($St >> 1$)
- **Bubble Dynamics** ($\Box \cdot u = 0$)
- **Stokesian Dynamics** ($Re << 1$)
  
  
  \[
  Re = \frac{\rho U a}{\eta} < 1, \quad St = \frac{\rho_i}{\rho} Re
  \]
  
  \[
  Pe = \frac{V}{D} Re
  \]

### Molecular Dynamics

\[
\frac{N_i}{N} \sim \left(\frac{a_i}{a} \right)^{1} \quad \frac{\tau_i}{\tau} \sim \left(\frac{a_i}{a} \right)^{0} \quad \text{GMU} \sim \left(\frac{a_i}{a} \right)^{c} N_i
\]
Simulation of two-phase matter

Bubble Dynamics

\[ Re = \frac{\rho_t \alpha^2 \gamma}{\eta} \]

Not physically relevant

\[ \rho_s < \rho_t \]

\[ \rho_s = \rho_t \]

Stokesian Dynamics

\[ Pe = \frac{\alpha^2 \gamma}{D} \]

\[ 1/Pe \]

Granular Dynamics

\[ Si = \frac{\rho_s D}{\rho_t \nu} \leq 1 \]
The End